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THE CANONICAL FORM OF ALL PLANAR LINKAGE TOPOLOGIES

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ABSTRACT

The current paper shows that all the planar linkages can be constructed from given components called, Assur Graphs, which can be ordered in a table with infinite numbers of rows and columns. In the paper we term this order the *canonical form* of the planar linkages. This canonical form is proved to be an ordered hierarchy of several levels, enabling systematic generation of all its members. The work has originated from the concept of Assur groups, long known in the field of kinematics, and used to decompose any linkage into basic kinematical atoms. In this paper we introduce a systematic procedure for generating all Assur groups thus finding all the topologies of plane linkages. The work in 2D is based upon known but new mathematical theorems which prove it to be complete and sound. The paper also indicates how this work can be extended into 3D linkages.
The mathematical foundation of this work contains several new theorems that have been published by the rigidity theory community during the past six years.

1. INTRODUCTION

This paper introduces a systematic methodology enabling derivation of the topologies of planar linkages. Although this generic perspective yields all planar linkage topologies, it is done using only two operations.
The idea behind the paper is based on the works of Assur (Assur 1952), but in a way different than he anticipated. The idea behind Assur's work was to decompose every linkage into Assur groups, based on the following theorem: for every linkage there is a unique decomposition into Assur groups. He used this theorem for analysis. The work, reported in the paper, is used for topological synthesis. The work here reports, for the first time in the engineering community, a method to construct all the Assur groups, by only two operations based on theorems from rigidity theory. Once we have all the Assur groups, different combinations between the Assur groups yield diverse topologies of linkages.
Thus, we now have the map of all the topologies of planar linkages.
The mathematical proof underlying this work is to be found in two papers published in the rigidity theory community. The first paper was published in 2003 by Berg and Jordan who proved that there is a set of graphs with a unique property, that possess a self-stress on all the edges, and all these graphs can be derived using only two operations (Berg and Jordan, 2003). They called these graphs – *generic cycles*. In 2009, a paper will be published in the European Journal of Combinatorics (Servatius et al., 2009a) in which one of the results shown, that for all the Assur groups, when their ground joints are identified, the resulting graph is a graph of the kind reported in 2003, known as generic cycles.
These two works are the mathematical foundation underlying this paper. The Assur groups were reformulated in terms of graphs, and using two operations similar to those reported in 2003 enable derivation of all the Assur groups.
Since the paper deals with engineering issues using material from rigidity theory, new definitions are introduced and Assur groups are treated and defined as Assur graphs.
The work ends with the possible extension into spatial linkage topologies. It seems that infinite topologies of spatial linkages with spherical joints can be derived using the methodology appearing in the paper, but, in contrast to 2D, it is not mathematically proved that all the topologies can be derived. Explanations appear in the paper.
Since such work, that is based on the theorems from rigidity theory, to the knowledge of the author, has not yet

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been published, and the paper relies only on the works of Assur and those published in the rigidity theory community, a comprehensive literature survey is not given. However, it is important to notice that many important works have been done in the mechanical engineering community related to syntheses and finding all the different topologies of mechanisms, including gear trains. The reader is referred to known works, such as: (Buchsbaum and Freudenstein, 1970), (Crossley, 1965), (Freudenstein and Dobrjanskys, 1965), (Dobrjanskys and Freudenstein, 1967) and more. One of the important books in this field, written by (Tsai, 2001) summarizes these works and introduces a systematic combinatorial approach for enumerating the kinematic structures of mechanisms.

2. OVERVIEW OF THE ASSUR GRAPH CONCEPT

Assur groups, occasionally referred as Assur Structures, are widely used in the kinematical community, particularly among Russian scientists. Leonid Assur (Assur, 1952) developed these basic structures in order to make it possible to decompose any linkage into components of zero mobility, and for each one, to develop special methods for analysis of locations, velocities, accelerations and other physical properties.

The concept has been reformulated for the first time in rigidity theory terminology in (Servatius et al., 2009a,b), where it was defined as a rigid graph, for which deletion of any vertex results in a non-rigid graph. Accordingly, it has been shown that an Assur Graph is a basic entity applicable to treatment not only of kinematical systems, but also static systems .

The current work employs Assur graphs as the central building block of the canonical form hierarchy. Since the paper deals only with the topology of planar linkages and all the mathematical foundation of this paper is graph theory, and the terminology from graph theory appearing in the paper can be found in any basic textbooks on the subject, such as (Swamy and Thulasiraman, 1981). For example, joints are referred to as vertices, links as edges and structures as graphs. Moreover, to avoid other terminologies used in the rigidity theory community and not in mechanical engineering the definitions appearing in the paper are slightly modified by giving them more physical than combinatorial meaning.

To clarify the terminology used in the paper let us define the structure depicted in Fig. 1 in both terminologies. In the terminology of engineering this is a determinate truss with four rods/bars, two joints – A and B, three pinned joints connecting rods 1,2 and 4 to the ground, while each rod has its specific geometry (length, inclination angle, etc.). Therefore, in engineering terminology there is a difference between the two determinate trusses in Fig 1.

In the terminology of rigidity theory the graph in Fig 1a is a rigid graph with four edges, two inner vertices, three ground vertices, three ground edges – 1,2 and 4 and there is no notion of geometry of the elements. Thus, from the rigidity theory point of view there is no difference between the two graphs in Figure 1.

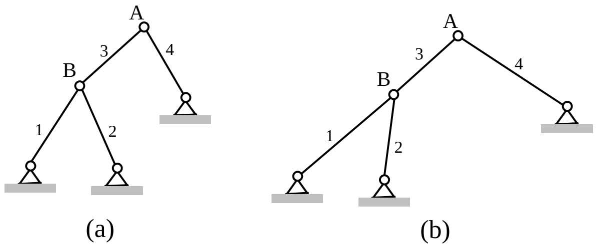

Figure 1. Two configurations with the same topology of a determinate truss (rigid graphs).

Now, we shall define Assur graphs and outline what distinguishes them from other rigid graphs.

Assur Graph – is a minimally rigid graph with e(G)=2*v(G) where e(G) and v(G) stand for the number of edges and inner vertices of graph G, respectively. The main property of the graph is that removal of any vertex with its incident edges makes the graph non-rigid.

Figure 2a depicts such an Assur Graph, while the graph in Figure 2b is not an Assur Graph since removing vertex A with its two incident edges, 3 and 4, results in a rigid graph – the dyad with the two edges 1 and 2.

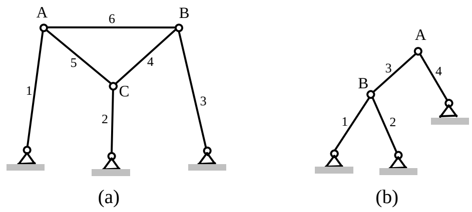

Figure 2. Different types of graphs in 2D
a) Assur Graph. b) Rigid graph that is not an Assur graph.

In each Assur Graph there are two types of vertices: ground, called also pinned vertices, and inner vertices.

For example, in triad type Assur Graph there are three inner and ground vertices while in the dyad type Assur graph there are two ground vertices and one inner vertex.

The composition rule for constructing a determinate truss from its components (Assur Graphs) is done as follows.

Let G_1 and G_2 be two Assur Graphs. G_1 is defined to be preceding G_2 if at least one ground vertex of G_1 is connected to an inner vertex of G_2.

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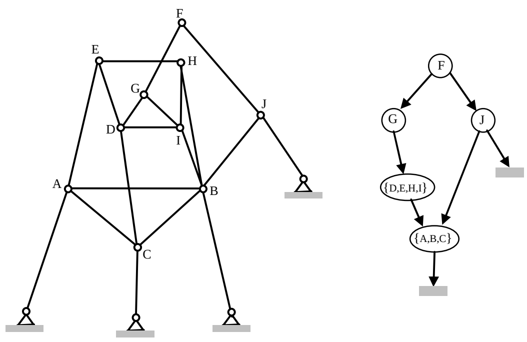
The decomposition process can be presented by a directed graph in which an edge e=<u,v> indicates that the Assur Graph corresponding to vertex u is preceding another Assur Graph, presented by vertex v. This type of graph is termed in the paper – *decomposition graph*.
For example in Figure 3.b the graph presents the order in which the determinate truss in Figure 3.a can be decomposed. We start with the initial vertex - a vertex to which no edge incident. In this example the initial vertex 'F' corresponds to the dyad with the inner vertex 'F' and the two edges 'FG' and 'FJ'. Once this dyad is removed it is possible to remove, independently the dyads G or J, and so forth.



(a) (b)

Figure 3 – Example of decomposition a determinate truss into Assur Graphs.
a) The determinate truss. b) The decomposition graph.

From the above it follows that once we have all the Assur Graphs it is possible to construct all different determinate trusses by composing different Assur Graphs, each time in a different order.
The transformation from determinate trusses into planar linkages is easy and is done by just augmenting a driving link, each time to a different ground vertex.
In Figure 4 we can see the three planar linkages in which the driving link is augmented each time to a different ground vertex.

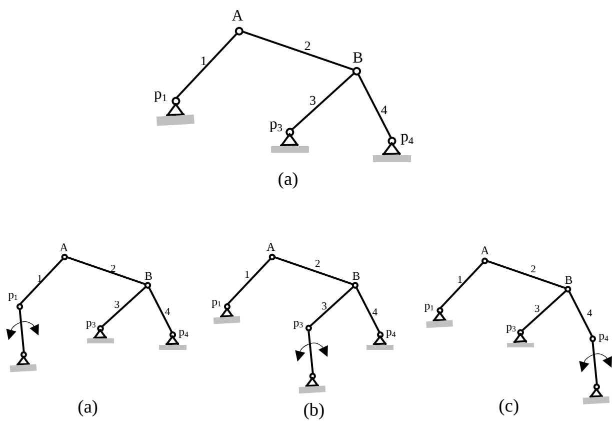


Figure 4. - Transformation of a determinate truss into planar linkages.

Now, that we know the definition of Assur Graphs, we can show how they are used in analysis of mechanisms. The idea of Assur, mathematically proven, is that every kinematical system has a unique decomposition into Assur Graphs. To clarify the decomposition we use a structural scheme (Figure 5) in which the driving link is deleted and replaced by a ground joint and each joint connects only two links. In the current example, link 1 is deleted and joint A is grounded. Then, the system is decomposed into three Assur Graphs, a tetrad, triad and a dyad. The order of the decomposition is important. If an inner joint of an Assur Graph, G1 becomes a ground vertex in Assur Graph G2, then G1 should precede G2.
The unique order of decomposition as appears in Figure 5 is: First analyze the tetrad{2,3,4,5}, and then analyze the Triad {6,7,8,11} or the dyad {9,10}, independently.

2.1 THE ATOMIC ASSUR GRAPH

For each dimension there corresponds a so-called atomic Assur Graph, which is the basis for the generation of the fundamental Assur Graphs, which in their turn, as will be explained in the next section, are the basis for the development of all Assur graphs. The atomic Assur Graph in 2D is the dyad.
Summing up, in this section it will be shown how all Assur Graphs are derived from the dyad by only two operations.

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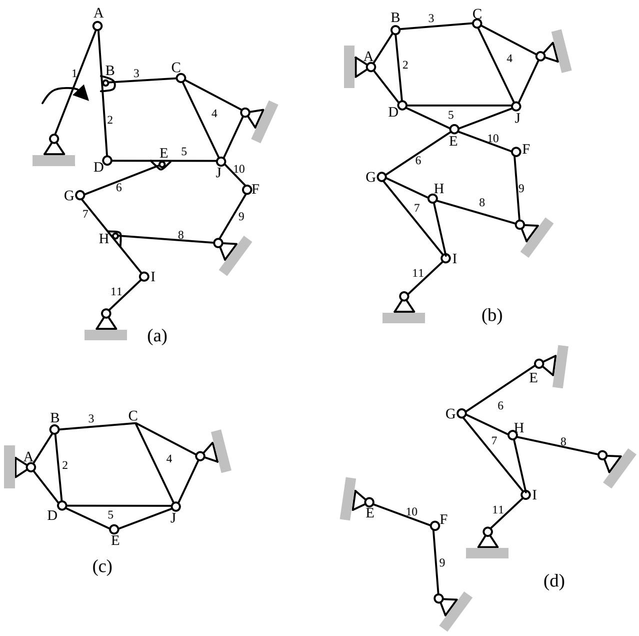

Figure 5. Example for decomposition of a linkage into Assur Graphs.
a) The linkage. b) The structural scheme. c) The first Assur Graph, the tetrad, in the decomposition order. d) The two Assur Graphs, dyad and triad, which can be decomposed simultaneously.

2.2 THE DERIVATION OF THE FUNDAMENTAL ASSUR GRAPHS FROM THE ATOMIC ASSUR GRAPHS

Each fundamental Assur Graph, is a representative of an infinite number of Assur Graphs and subsequently of an infinite number of determinate trusses and linkages. Any feasible fundamental Assur Graph can be obtained through an application of a sequence of operations, termed – fundamental extensions, starting from the atomic Assur Graph of the corresponding dimension, as is described in detail in the following section. It is to be emphasized that these operations are invariant, i.e., the special properties of Assur graphs that exist at the singular positions that are reported and proved in (Servatius et al., 2009a), remain in the extension operations.

2.3 THE FUNDAMENTAL EXTENSION OPERATION IN 2D

The main property of the fundamental extension operation, transfers fundamental Assur Graphs into extended fundamental Assur Graphs, which contain an additional basic triangle.

The fundamental extension is applied to the ground edges of the fundamental Assur Graphs, namely, the edges incident to the ground vertices**.**

The fundamental extension is done in three stages: first, one ground edge is removed; then a basic triangle is added with one vertex coinciding with a non-ground vertex of the removed edge; then the remaining two free vertices of the basic triangle, are both connected to the ground through one ground edge.

In Figure 6 we can see the process of deriving the triad from the dyad by replacing one of its ground edges (in the dyad both edges are ground edges), designated by the bold line, yielding a triangle and an additional two ground edges. The following fundamental Assur graph is derived by replacing the ground edge whose end vertex is c resulting in the new fundamental Assur graph appearing in Figure 6c.

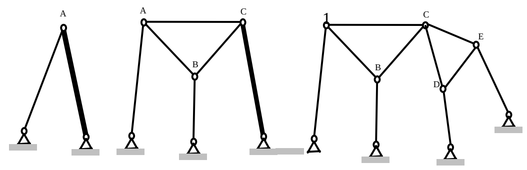

Figure 6. Example of applying the fundamental extension yielding fundamental Assur graphs. The dyad. b) The triad. c) The double triad.

3. THE CANONICAL FORM OF THE ASSUR GRAPHS

In the previous section we have introduced the canonical form of the fundamental Assur Graphs, being derived from the atomic Assur Graph, the dyad, through repetition of the single operation of fundamental extension. In this section, it will be shown that each fundamental Assur Graph is the representative of an infinite class of Assur Graphs, all derived through one operation, called the 1-extension operation, applied sequentially to the corresponding fundamental Assur Graph. In set theory terminology, there are an infinite number of classes of Assur Graphs, each class of which is defined by a representative graph – the corresponding fundamental Assur Graph. The following section explains how all the Assur Graphs of the class are obtained by applying the 1-extension to the representative Assur Graph. The table appearing in Figure 7 provides a general perspective on the canonical forms of all the Assur Graphs, as infinite classes (columns).

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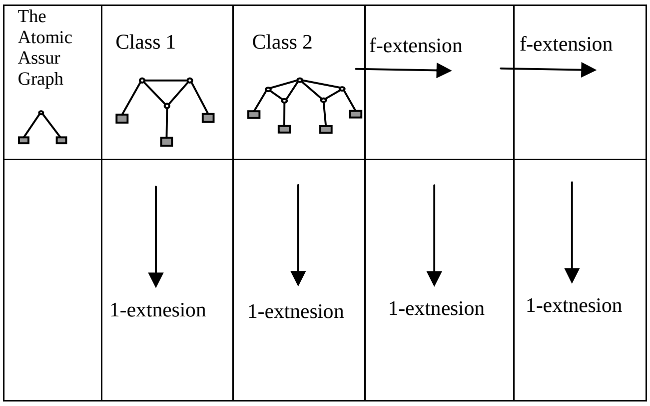

Figure 7. General perspective on the class hierarchy of the Assur Graphs

3.1 THE 1-EXTENSION OPERATION AND ITS APPLICATION RULE

In the previous section it was mentioned that each fundamental Assur Graph defines a class of an infinite number of Assur Graphs, all derived through a single operation called the 1-extension.

Definition: 1-extension. Choose an edge xy. Add an additional vertex z to the graph. Replace the edge xy by two edges zx, zy and an additional edge connecting vertex z with any other vertex of the graph.
Figure 8 schematically depicts the application of the 1-extension.

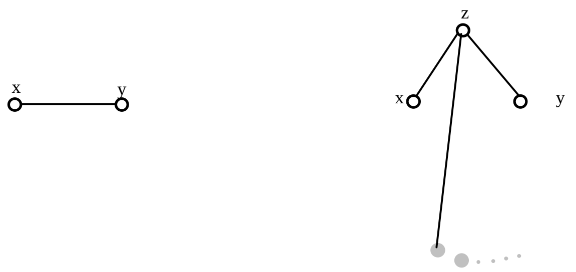

Figure 8. The 1-extension operation applied on an edge.

The special case of the 1-extension in two dimensions is widely used and reported in the literature (Berg and Jordan, 2003). This operation is invariant, i.e., the special properties of Assur graphs at their singular positions remain in the extension operation.

3.2 OBTAINING THE CANONICAL MAP OF ASSUR GRAPHS IN 2D

The canonical form of Assur Graphs consists of fundamental Assur Graphs that are the representatives of an infinite number of Assur Graphs, such that each Assur Graph belongs to one and only one class. The first column, whose representative Assur Graph is the dyad, is different from other columns (as is shown in Table 1), since there are no elements in its class except for the dyad itself.

4 CHARACTERISATION AND IDENTIFICATION OF FUNDAMENTAL ASSUR GRAPHS

It is possible to assign an ID to each fundamental Assur Graph that can be used to identify it and also from which it is also possible to construct it. All the fundamental Assur Graphs can be sorted according to the lexicographic order, using the following rule:

1. Find the longest chain in the fundamental Assur Graph, which will define the number representing the main chain. The ordering in the main chain is defined so that the number obtained will be the largest in the lexicographic order.
2. This numbering is defined for every sub-chain, recursively.

For example, there are two possibilities for numbering the fundamental Assur Graph in Figure 9a as shown in 9b and 9c. The corresponding number for the framework in Figure 9b is 1231456 and for the second framework is 1234_156. Since $123_1456 > 1234_156$ in lexicographic order, thus the numbering in Fig. b will be the one chosen.

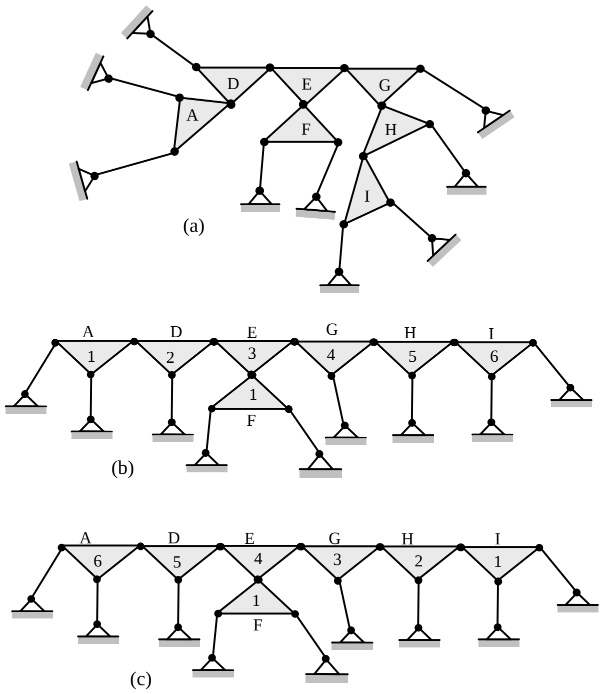

Figure 9. Example of different numbering of fundamental Assur Graphs.

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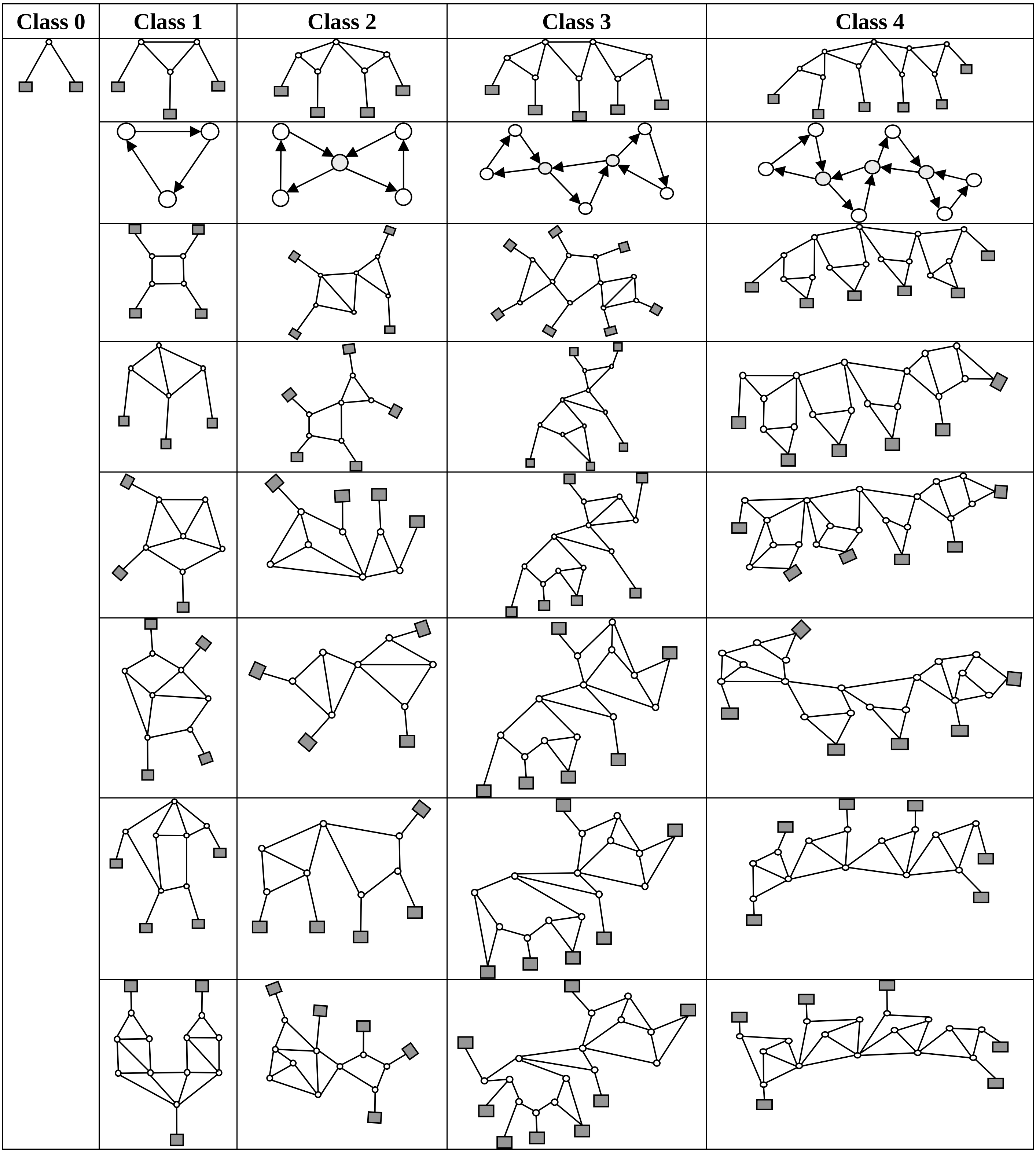
Table 1 - Some classes of Assur graphs in 2D



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An additional example is shown in Figure 10, with the fundamental Assur Graph associated with the unique numbering of: $1 2 3_1 4 5_1 6_1 7_1 8 9 (10)$.

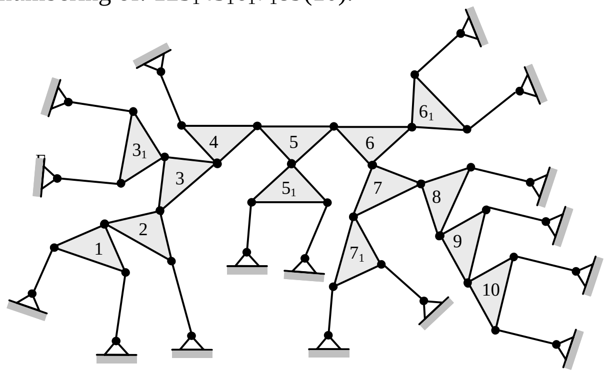

Figure 10. Example for a fundamental Assur Graphs, whose unique numbering is $1 2 3_1 4 5_1 6_1 7_1 8 9$ 10).

5. THE MATHEMATICAL FOUNDATION UNDERLYING THE PROOF OF THE COMPLETENESS AND SOUNDNESS OF THE 2D ASSUR GRAPH TABLE

In this section we show that the map of all the Assur Graphs in 2D is complete and sound, i.e,. all the Assur Graphs can be derived through the above two operations and all the graphs that are derived by applying the two operations are Assur Graphs. The mathematical foundation underlying this proof is based on two works reported in the rigidity theory literature.

The first work was published in 2003 (Berg and Jordan, 2003) who proved that there exists an infinite set of graphs with e=2v-2 possessing one self-stress on all the edges, i.e. internal forces on all the edges that satisfy the force equilibrium around each vertex. The type of their graphs is slightly different from Assur Graphs since they do not have ground vertices. Graphs that relate to trusses but do not have ground vertices are termed – frameworks, but still the main concern is rigidity of the frameworks. Thus, in this new terminology they defined frameworks with e=2v-2 that possess a unique self-stress on all the edges. They called these types of frameworks with the above property related to self-stress as - *generic cycles*. In Figure 11 are shown two frameworks with the same number of edges and vertices but only the one appearing in Figure 11b is a generic cycle since it possesses a unique self-stress while the framework in Figure 11a does not.

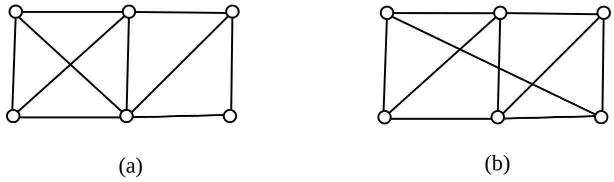

Figure 11. Example of two frameworks.
a) Not a generic cycle. b) Generic cycle

In their work, Berg and Jordan proved that it is possible to derive all the generic cycles from only one framework, a framework with four vertices, six edges and complete, i.e., there is an edge between any two vertices. The latter graph is designated in the literature – as K4 and is shown in Figure 12a. They have used two operations: the 1-extension and the 2-sum. The first operation is used in the current paper as well, while the latter is similar to the logical exclusive OR - XOR operation: you join two generic cycles by deleting a common edge and the remaining edges constitute the result of the 2-sum as shown in Figure 12b.

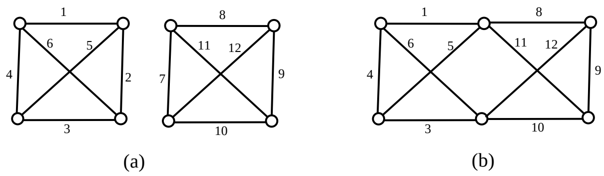

Figure 12. The 2-sum of two frameworks of type K4.
a) The two frameworks K4. b) The resulting framework.

The second work that the proof is based on is (Servatius et al., 2009a,b) which established the relation between Assur Graphs and generic cycles. In the latter paper it was proved that every Assur Graph corresponds to a generic cycle by contracting all of its ground vertices into one vertex. This latter graph is termed a *contracted Assur Graph*. In Figure 13a appears a triad, for which contracting its three ground vertices results in the known complete graph, K4.

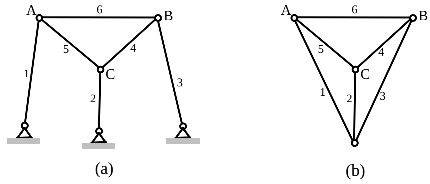

Figure 13. Transforming an Assur Graph into a generic cycle (contracted Assur Graph).
a) The triad. b) The corresponding contracted Assur Graph – K4.

It can be easily verified that all the Assur Graphs

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appearing in table 1 can be reformulated in the terminology of generic cycles. Thus, since for the latter it was proved that the two operations guarantee completeness and soundness thus it is valid also for all Assur Graphs.

In the next section we introduce the extension of this canonical form into 3D. In contrary to 2D, it is, yet, impossible to prove the completeness conjecture, i.e., there are still Assur Graphs in 3D that cannot be derived from the fundamental Assur Graphs.

6. THE CANONICAL FORM OF THE ASSUR GRAPHS IN 3D

The transformation from Assur Graphs in 2D into 3D, is straightforward. We start again with the dyad, this time consisting of three ground edges instead of two, as shown in Fig. 12a. In the fundamental extension we replace each ground edge with a triangle, but this time two edges come out from each of the two new vertices, as shown in Fig. 14b where a 3D triad was constructed through a fundamental extension from the 3D dyad.

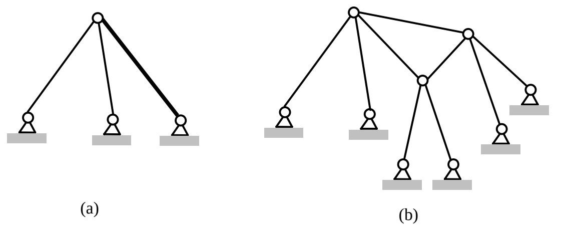

Figure 14. Example of a fundamental extension in 3D.
a) The 3D dyad. b) The 3D triad.

The idea of the classes, and that each fundamental Assur Graph is the representative of a class of Assur Graphs is the same as in 2D, only this time the operation transforming one Assur Graph into the successor is done through 2-extension. In the 2-extension the vertex that is added is connected this time to two other vertices and not to one as is done in the 1-extension. Example of deriving an Assur Graph from fundamental Assur Graphs appears in Figure 15 where the edge being split is indicated by the bold edge.

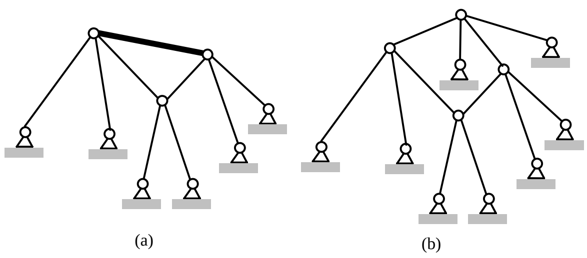

Figure 15. Example of applying the 2-extesnion in a 3D Assur Graph.
a) The fundamental Assur Graphs spatial triad. b) The resulting Assur Graph after applying the 2-extension.

The structure of the canonical form is the same as for the 2D Assur Graph. The first column consists of the spatial dyad, which does not have a correspondence in the 3D Assur Graph, thus there are no derivations in that column. The first row consists of the fundamental Assur Graphs, and each column contains all the derivations from that representative using 2-extension operation.

6.1 THE MATHEMATICAL PROOF UNDERLYING THE 3D ASSUR GRAPH

In contrast to 2D, there is no mathematical proof for the completeness of the 3D Assur Graphs. The main reason for that is that there are still mathematical problems that have not yet been resolved by the mathematicians in the rigidity theory community. Among these is the Assur Graph, appearing in Figure 16, for which there is no derivation from any fundamental Assur Graphs. The main problem is that the degree of each vertex is at least five.

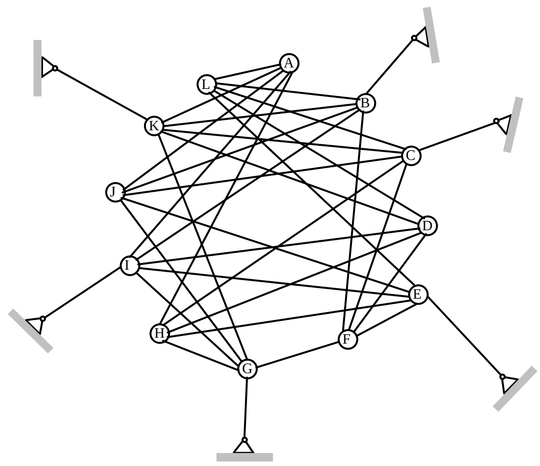

Figure 16. Example of a 3D Assur Graph for which there is no derivation from any fundamental Assur Graphs.

7. CONCLUSIONS AND FURTHER RESEARCH

The paper shows that there is a nice order in building all the topologies of planar linkages. All starts from the most basic topology - the dyad. Then through only two operations all the Assur Graphs are produced. Moreover, all Assur Graphs are arranged in a very nice order. There is infinite number of fundamental Assur Graphs derived by applying a sequence of fundamental extensions upon

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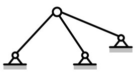
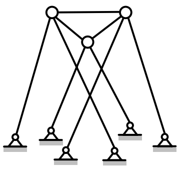
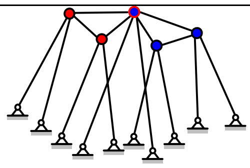
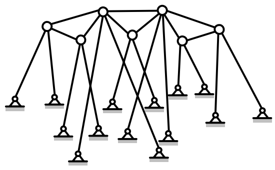
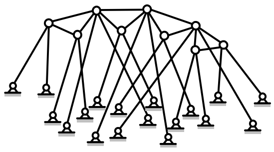
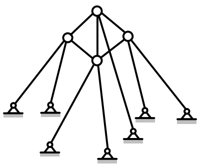
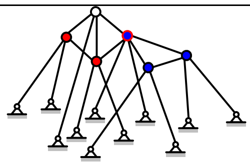
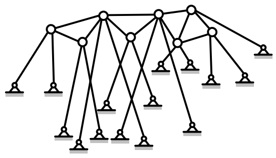
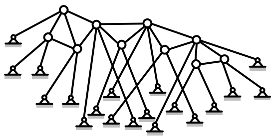
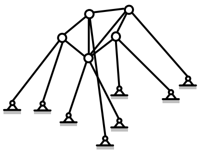
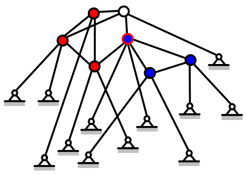
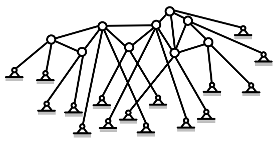
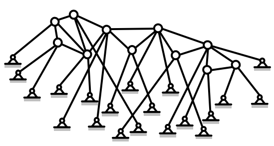
the dyad. If we arrange all the Assur Graphs in a table, each fundamental extension creates another column in this table. Now that we have infinite number of columns of the table, from each fundamental Assur Graph there is an infinite number of Assur Graphs derived by applying the 1-extension operation several times. In short, all the Assur Graphs are arranged in a table with infinite columns and infinite rows and each Assur graph in the table has a specific sequence of applying the mentioned two operations. It should be noted that it was proven in the references quoted that the two operations preserve the combinatorial properties of Assur Graphs.

Once we have all the Assur Graphs, when we connect several Assur graphs by connecting ground joints of one graph into inner joints of the other and adding driving links we obtain the topology of all the linkages.

In 2D it is mathematically proved that all the Assur Graphs appear in the above table. This is not the case in 3D. There are still Assur Graphs that cannot be derived from the spatial dyad. Once a mathematical variant of 2-extension will be revealed in the rigidity theory community we will have all the topologies of all linkages in 3D.

The mathematical basis of the validity of the presented technique has been omitted from this paper, as it has been widely elaborated in previous publications of the author in collaboration with the discrete mathematical community.

Table 2. The table of 3D Assur Graphs

Class 1	**Class 2**	**Class 3**	**Class 4**	**Class 5**
				
				
				

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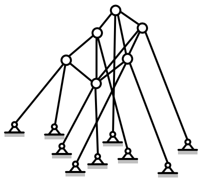
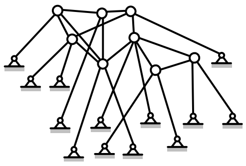
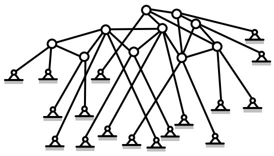
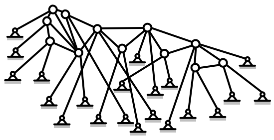
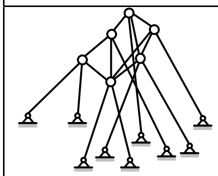
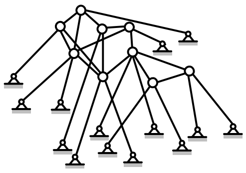
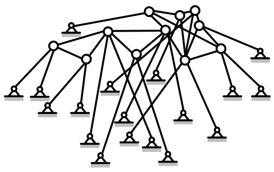
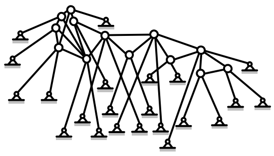
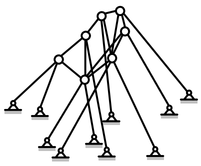
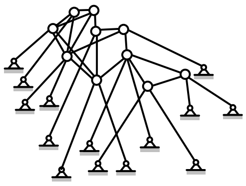
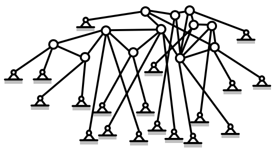
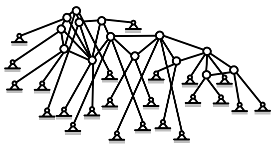
Class 1	Class 2	Class 3	Class 4	Class 5
				
				
				

8. REFERENCES

Assur, L.V., 1952, Issledovanie ploskih sterzhnevyh mehanizmov s nizkimi parami s tochki zrenija ih struktury i klassifikacii. Akad. Nauk SSSR, Edited by I.I. Arotobolevskii

Berg A. R. and Jordan T., 2003, A proof of Connelly's conjecture on 3-connected circuits of the rigidity matroid, *J. Combin. Theory Ser. B,* 88(1), pp. 77-97.

Buchsbaum F. and Freudenstein F., 1970, Synthesis of Kinematic Structure of Geared Kinematic Chains and other Mechanisms, Journal of Mechanisms, 5, pp. 357-392.

Crossley F.R.E., 1965, The Permutation of Kinematic Chains of Eight Members or Less from the Graph-Theoretic Viewpoint, in *Developments in Theoretical and Applied Mechanics*, Vol. 2, Pergamon Press, Oxford, pp. 467-486.

Dobrjanskyj L. and Freudenstein F., 1967, Some Application of Graph Theory to the Structural Analysis of Mechanisms, *ASME Journal of Engineering for Industry*, Series B, 89, pp. 153 – 158.

Freudenstein F. and Dobrjanskyj L., 1965, On a Theory for the Type Synthesis of Mechanism, in *Proceedings of the 11th International Congress of Applied Mechanics*, Springer, Berlin, 420-428.

Tsai L.W., 2001, *Enumeration of Kinematic Structures According to Function*, CRC Press, Boca Raton.

Servatius B., Shai O. and Whiteley W., 2009a, Combinatorial characterization of the Assur graph from engineering, accepted for publication in the *European Journal of Combinatorics..*

Servatius B., Shai O. and Whiteley W., 2009b, Geometric properties of Assur graphs, accepted for publication in the *European Journal of Combinatorics*, .

Swamy MN, Thulasiraman K., 1981, Graphs: networks and algorithms. New York: Wiley.

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